- Logic is concerned with properties of statements that hold by virtue of form alone regardless of content
- Premises (logically) imply or have as a (logical) consequence a conclusion because in any argument of the same form, if the premises are true, then the conclusion is true.

1 Syntax

Vocabulary

1. Logical Symbols
   (a) **Connectives**: $\neg, \vee, \wedge$
   (b) **Individual variables**: $v_0, v_1, \ldots$
   (c) **Quantifiers**: $\forall, \exists$
   (d) **Identity**: $=$, a 2 place predicate
2. Non-Logical Symbols
   (a) **Constants or individual symbols**: $f_1^0, f_2^0, f_3^0, \ldots$(denumerable)
   (b) **k-place predicates or relation symbols**:

$$\begin{array}{cccc} A_0^1 & A_1^1 & A_2^1 & \ldots \\ A_0^2 & A_1^2 & A_2^2 & \ldots \\ A_0^3 & A_1^3 & A_2^3 & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

   (c) **Function symbols**: (n and k are any positive integers):

$$\begin{array}{cccc} f_0^1 & f_1^1 & f_2^1 & \ldots \\ f_0^2 & f_1^2 & f_2^2 & \ldots \\ f_0^3 & f_1^3 & f_2^3 & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Dfn **1.1.** *Language: an enumerable set of non-logical symbols*

- Any language is a subset of this vocabulary.
- *Empty language ($L_\varnothing$)*: no non-logical symbols.
  This language has infinitely many formulas because formulas can be constructed with identity (a logical symbol) and quantifiers. $(\forall x)(x = x)$, $(\forall x)(x = x \vee (x = x)$, and so on.
- *Language of Arithmetic (L^*)*:
  individual constant: f_4^0 (say) for 0
  two-place predicate: A_0^2 for $<$
  one-place function symbol: f_1^2 for $'$
  the two-place function symbols: f_1^3,f_2^3 for $+,\bullet$

Dfn **1.2.** *Term: atomic term or built up from atomic terms in a sequence of finitely many steps (in a* formation sequence*) by applying function symbols to simpler terms*

- *Atomic terms*: all individual variables or individuals constants are terms.
- *Formation Rules*: if f is an n-place function symbol and $t_1, \ldots t_n$ are terms, $f(t_1, \ldots, t_n)$ is a term

Dfn **1.3.** *Formulas: A (first-order) formula is anything that is an atomic formula or can be build up from atomic formulas in a sequence of finitely many steps by applying formation rules.*

- *Atomic Formula* (for): a string of symbols $R(t_1, \ldots, t_n)$ consisting of a predicate, followed by a left parenthesis, followed by n constants or variables, where n is the number of places of the predicate, with commas separating the successive terms, all followed by a right parenthesis
- *Formation Rules*: (- means 'followed by')

(F-Neg) if F is a formula, '$\neg$' $-$ F is a formula
(F-Disj) if F and G are formulas, '(' $-$ F $-$' $\vee'$ $-$ G $-$ ')', '(' $-$ F $-$' $\wedge'$ $-$ G $-$ ')' are formulas.
(F-Quan) if F is a formula and x is a variable, '$\forall$'$-x-$ F, '$\exists'$ $- x -$ F are formulas.
(F-Ident) If t_1 and t_2 are terms, $= (t1, t2)$ is a formula.

Some more notions

Dfn 1.4. *subterm*: A consecutive string of symbols inside a term is a *subterm* if it is itself a term.

Dfn 1.5. *formation sequence* A sequence of finite steps by applying formation rules.

Dfn 1.6. A consecutive string of symbols inside a formula is a *subformula* if it is itself a formula.

Dfn 1.7. Terms that contain variables are *open* terms, while terms that do not are *closed* terms.

Dfn 1.8. A formula is a **sentence** if it contains no free variables.

Dfn 1.9. A **subsentence** is a subformula that is a sentence.

Rem 1.10. A formula need not be bound, and the variables associated with quantifiers need not appear in the formula. $(\forall x)Fx$, $(\forall x)Fxy$, $(\forall y)Fxy$, $(\forall z)Fxy$, $(\forall x)(\forall y)Fxy$

Abbreviations in Logic

Symbol	Official	Unofficial
$<$	$<(x, y)$	$x < y$
P	$P(x, y)$	Pxy
$\wedge$	$(A \wedge (B \wedge (C \wedge D)))$	$(A \wedge B \wedge C \wedge D)$
$\rightarrow$	$(\neg G \vee F)$	$(F \leftrightarrow G)$
$\leftrightarrow$	$((\neg G \vee F) \wedge (\neg F \vee G))$	$(F \rightarrow G)$
$\forall y < x$	$\forall y(y < x) \rightarrow \ldots)$	$\forall y < x$
$\exists y < x$	$\exists y(y < x) \rightarrow \ldots$	$\exists y < x$

Abbreviations in Language of Arithmetic (L*))

Official	Unofficial
v_0	x
f_0^0	0
$f_0^1(f_0^0)$	1
$f_0^1(f_0^1(f_0^0))$	2
$f_1^2(f_0^1(f_0^1(f_0^0)), v_0)$	$2 \bullet x$
$f_0^2(f_1^2(f_0^1(f_0^1(f_0^0)), v_0), f_1^2(f_0^1(f_0^1(f_0^0)), v_0))$	$2 \bullet x + 2 \bullet x$

2 Syntactic Properties

Lem **2.1.** *Parenthesis Lemma (BJ-L9.4): When formulas are written in official notation the following hold:*

1. *Every formula ends in a right parenthesis.*
2. *Every formula has equally many left and right parenthesis.*
3. *If a formula is divided into a left part and a right part, both nonempty, then there are at least as many left as right parentheses in the left part, and more if that part contains at least one parenthesis.*

Lem **2.2.** *Unique Readability Lemma (BJ-L9.5)*

1. *The only subformula of an atomic formula $R(t_1, \ldots, t_n)$ or $= (t_1, t_2)$ is itself.*
2. *The only subformulas of $\neg F$ are itself and the subformulas of F*
3. *The only subformulas of $(F \wedge G)$ or $(F \vee G)$ are itself and the subformulas of F and G.*
4. *The only subformulas of $\forall xF$ or $\exists xF$ are itself and subformulas of F*

Lem **2.3.** *(BJ-P9.4) In a formation sequence for a formula F, every subformula of F must appear*

Lem **2.4.** *(BJ-P9.5) Every formula F has a formation sequence in which the only formulas that appear are subformulas of F, adn the number of formulas that appear is no greater than the number of symbols in F*