Dfn **0.1.** *Terminology*

- *'theorem': most important general results*
- *'lemmas' : results that are steps on the way to a theorem*
- *'corollaries': results that follow directly from some theorem*
- *'propositions': results that are proven but do not follow from an important theorem*
- *'examples': particular results rather than general*

1 | Cantor's Thm and Two Methods of Proof

Not all sets are enumerable: some are bigger.

(Thm.1.1) **(Cantor's Theorem)**. The set of all sets of positive integers is not enumerable.

Proof by Informal Method:
(Dfn.1.2) set Δ(L) iff for each positive integer n, (n is in Δ(L) iff n is not in S_n, where an infinite list L $S_1, S_2, S_3 \ldots$ of sets of positive integers,
E.g. $\{\{1,2\}\{1,3,2\}\}$; $\Delta(L) = \emptyset$;
E.g. $\{\{5\},\{2\},\{6,6\}$; $\Delta(L) = 2$

Sub-Proof: Show by reductio that the set Δ(L) is never in the given list L

- Suppose that Δ(L) does appear somewhere in list L as entry m i.e. for some positive integer m, $S_m = \Delta(L)$
  E.g. if 127 is such an m, we are supposing that Δ(L) and $S_1 27$ are the same set i.e. that a positive integer belongs to Δ(L) iff it belongs to the 127th set in list L.
- Apply definition to m: with n = m, (Dfn1.2) tells us that m is in Δ(L) iff m is not in S_m
- A contradiction follows: if S_m and Δ(L) are one and the same set we have m is in Δ(L) iff m is in S_m. contradiction
- Applied to any list of sets of positive integers, the dfn yields a set of positive integers which was not in the list. Then no list enumerates all sets of positive integers: the set of all such sets is not enumerable.

- *General Strategy: Informal Method*:
  1. for a list L of sets of positive integers, define set Δ(L) of positive integers not named in L
  2. Prove that if try to add Δ(L) to the list as a new first member, the same method, applied to new list L* will yield a different set Δ(L*) that is likewise not on the augmented list.+

Proof by Diagonalization with Functions:

- Represent sets $S_1, S_2, \ldots$ by functions $s_1, s_2, \ldots$ of positive integers take the numbers 0 and 1 as values. The relationship between the set S_n and the corresponding function s_n: for each positive integer p we have

$$s_n(p) = \begin{cases} 1 & \text{if} p \text{ is in } S_n \\ 0 & \text{if } p \text{ is not in } S_n \end{cases}$$

- Visualization of list as array of 0s and 1s, where the nth row represents the function s_n and so set S_n

	1	2	3	4	5	...
s_1	$s_1(1)$	$s_1(2)$	$s_1(3)$	$s_1(4)$	$s_1(5)$	...
s_2	$s_2(1)$	$s_2(2)$	$s_2(3)$	$s_2(4)$	$s_2(5)$	...
s_3	$s_3(1)$	$s_3(2)$	$s_3(3)$	$s_3(4)$	$s_3(5)$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮...

- Sequence: the nth row $s_n(1), s_n(2), s_n(3), s_n(4), \ldots$ is a sequence of zeros and ones in which the pth entry, $s_n(p)$, is 1 or 0 according as the number p is or is not in the set S_n.

- **Diagonal Sequence**: entries (0s/1s) in the diagonal of the array: $s_1(1)s_2(2)s_3(3)s_4(4)\ldots$
- **Diagonal Set**: the set of integers determined by zeros and ones in the diagonal sequence.
- The diagonal set may well be among those listed in L. In other words, there may well be a positive integer d such that the set Sd is none other than our diagonal set. The sequence of zeros and ones in the dth row of Figure 2-1 would then agree with the diagonal sequence entry by entry: $sd(1) = s1(1), sd(2) = s2(2), sd(3) = s3(3), \ldots$. That is as may be: the diagonal set may or may not appear in the list L, depending on the detailed makeup of the list. What we want is a set we can rely upon not to appear in L, no matter how L is composed.
- **antidiagonal set**: set consisting of positive integers not in the diagonal set i.e. the sequence is obtained by changing zeros to ones and ones to zeros in the diagonal sequence i.e. $1-s_1(1), 1-s_2(2), 1-s_3(3), 1-s_4(4), \ldots$.

  Proof: the Antidiagonal does not appear in table

- If it did appear — say, as the mth row — we should have $sm(1) = 1s_1(1), sm(2) = 1s_2(2), \ldots, sm(m) = 1s_m(m), \ldots$ But the mth of these equations cannot hold.
- Proof: $s_m(m)$ must be zero or one. If zero, the mth equation says that 0 = 1.
- If one, the mth equation says that 1 = 0.
- So, antidiagonal sequence differs from every row of our array
- So, antidiagonal set differs from every set in our list L

(Cor.1.3) The set of real numbers is not enumerable.

Proof:

- If ζ is a real s.t. $0 < \zeta < 1$, then ζ has a decimal expansion $.x_1x_2x_3\ldots$ where each x_i is one of the cyphers 0–9.
- Some numbers have two decimal expansions, since for instance .2999. . . = .3000. . . ; so if there is a choice, choose the one with the 0s rather than the one with the 9s.
- Then associate to ζ, the set of all positive integers n such that a1 appears in the nth place in this expansion.
- Every set of positive integers is associated to some real number (the sum of 10−n for all n in the set), and
- so an enumeration of the real numbers gives rise to an enumeration of the sets of positive integers, which cannot exist, by the preceding theorem.

  Theorem. the set of real numbers is not countable.

Proof: Suppose otherwise, that is, there is a sequence rn which exhausts the real numbers. Then we can choose a closed interval [a1,b1] which avoids r1. Next, choose a sub-interval [a2,b2]⊂[a1,b1] which avoids r2. Continue in like manner to obtain a decreasing sequence of intervals [an,bn] such that for all n, [an,bn] avoids rn. Now, the an form an increasing sequence, bn a decreasing sequence, and for all n we have an≤bn. It follows that supan≤infbn and in particular, there exists a point x∈[supan,infbn]. Then for all n we have x≠rn, since x∈[an,bn] and we said that [an,bn] avoids rn. But this contradicts the exhaustive nature of rn.□

2 | Enumeration without Practical Limitations

the members of an infinite set cannot be arranged by anyone into a list: not a finite creature (human or machine), not enough time or paper to write it on. They can be (thought of as) arranged by an infinite being.

Zeus Enumeration: If a set is enumerable, Zeus can enumerate it in one second by writing out an infinite list faster and faster. He spends 1/2 second writing the first entry in the list; 1/4 second writing the second entry; 1/8 second writing the third; and in general, he writes each entry in half the time he spent on its predecessor. At no point during the one-second interval has he written out the whole list, but when one second has passed, the list is complete. (See Figure)

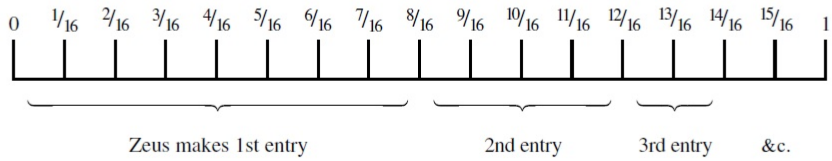

Figure 2-2. Completing an infinite process in finite time.

Zeus could produce a genuine list which exhausts the entries in all the lists, by using some such device as we used in the preceding chapter to enumerate the positive rational numbers. Nevertheless, Cantor's diagonal argument shows that neither this nor any more ingenious device is available, even to a god, for arranging all the sets of positive integers into a single infinite list. The impossibility of enumerating all the sets of positive integers is as absolute as the impossibility of drawing a round square, even for Zeus.