

1 | Enumerability

- What is an infinite set? One kind can be defined in terms of enumerability
- function = function of positive integers, leaving it open whether the function is total or partial (Dfn..1) **Informal:** A set S is **enumerable** iff S is a set that can be enumerated i.e. arranged in a single list with a first entry, second, and so on, so that every member of the set appears sooner or later on the list i.e. as the n th entry, for some finite n .
- *What is being arranged?* The entries in these lists are not numbers but numerals, or names of numbers. In general, in listing the members of a set you manipulate names, not the things named. For instance, in enumerating the members of the United States Senate, you don't have the senators form a queue; rather, you arrange their names in a list, perhaps alphabetically.
- A list that enumerates a set can be finite or unending.
- An infinite set that is enumerable is **enumerably infinite**; aka. **denumerable, countable**
- Adequacy Conditions for a legitimate list
 1. each member show up at least once i.e. it must be complete
 2. every element of the set being enumerated be associated with some positive integer, not that every positive integer have an element of the set associated with it.
permitted: redundant lists with repeat set members; lists with gaps (we can erase them)
- Examples of illegitimate lists:
positive integers: 1, 3, 5, ... 2, 4, 6, ...;
- (Dfn..2) **Formal - 3 equivalent definitions:** S is enumerable iff 1) there is an (onto, not necessarily 1-1, total or partial) function $f: P \rightarrow S$ whose range is all of S ; 2) there is a function all of whose arguments are positive integers and all of whose values are members of S , and that each member of S is a value of this function: For each member a of S there is at least one positive integer n to which the function assigns a as its value.; 3) A set is enumerable if and only if it is the range of some function of positive integers.
- More precise informal definition of enumeration of S : an arrangement in which each member of S is associated with one of the positive integers. The thing named by the first entry in the list is associated with 1, and thing named by the n th entry is associated with n .
- an infinite list **determines** a function (call it f) that takes positive integers as arguments and takes members of the set as values.
E.g. the set 2, 4, 6, 8, ... which enumerates the set E of even positive integers determines the function f for which we have $f(1) = 2, f(2) = 4, f(3) = 6, f(4) = 8, f(5) = 10, \dots f(n) = 2n$ determines the set of positive integers; $f(n) = 2n$ enumerates the list of positive of integers.
- **Identity Function:** enumerate the set of all positive integers P by the function that assigns to each positive integer n the value n itself. $id(n) = n$ for each positive integer n .
- **Multiple Functions:**
If one function enumerates a nonempty set, so do infinitely many others. the set of positive integers is enumerated not only by the function id , but also by the function (call it g) determined by the following list: 2, 1, 4, 3, 6, 5, ..., a list that interchanges entries of original list. $g(n) = -n + 1$ if n is odd; $n - 1$ if n is even.
- **Partial Functions:** gappy list: 1, -, 2, , 3, , 4, , ...
 $h(n) = (n + 1)/2$; if n is odd; undefined otherwise partial function h is a strange but perfectly acceptable enumeration of the set P of positive integers.
- **Easy enumeration with partial functions:** e.g. the set E of even integers is enumerated by $j(n) = n$ if n is even; otherwise undefined; Any set S of positive integers is enumerated simply by: $s(n) = n$ if n is in the set S ; undefined otherwise.
- **Encoding/Decoding Functions:** an encoding function $G: S \rightarrow P$ is a 1-1 function that assigns a positive integer to each member of S . A decoding function $g: P \rightarrow S$ is any function that enumerates S . If there is an encoding function G , then there will also be a decoding function, namely the inverse of G . Sometimes the best way to show a set is enumerable is to give an encoding function, rather than a decoding function.

2 | Examples

1. \emptyset is enumerable.
It is stipulated to be enumerable i.e. the list with no members
 $f = \emptyset$.
2. any finite set is enumerable e.g. $\{1, 2\}$
1, 2
 $f(n) = n$ if $n = 1, 2$; otherwise undefined.
3. the set of all people on earth is enumerable.
any finite non-empty set, by definition, can be written as $S = \{s_1, \dots, s_n\}$ for some positive number n . But this is just to say that there is a function from P to S , namely, $f(k) = s_k$ if $k \leq n$, and undefined otherwise.
4. The set of positive integers
1, 2, ...
 $f(n) = n$
5. the set of odd numbers:
1, 3, 5, 7, ...
 $f(n) = 2n - 1$
6. The set of integers
0, -1, 1, -2, 2, -3, 3, -4, 4, ...
7. Ordered pairs of positive integers: (1, 1), (1, 2), ..., (2, 1), (2, 2), ...

	1	2	3	4	5	...
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	...
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	...
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	...
...						...

Solutions:

- (a) Show an enumerable list: from top-left, follow diagonally. (1,1),(2,1)(1,2), ...
- (b) Show an enumerable list: use up every other place in listing the pairs (1,n), every other place then remaining in listing the pairs (2, n), and so on.
(1,1),(2,1),(1,2),(3,1)(1,3),(2,2),(1,4),(4,1),(1,5),(2,3),...
- (c) Encoding function:
decoding function Expanding square-fronts: traverse each backwards L (with corner (n, n)).
- (d) Method 3: Place each series in alternating slots that remain vacant (see text).
- (e) Write down the function $G(m, n)$ that encodes (m, n) on methods 1 and 3.
8. Any subset S of P is enumerable.
Use $f(n) = n$, $n \in S$; undefined otherwise. d) φ is enumerable.
We can use the partial function e , whose domain is empty: undefined everywhere! Then the range of e is φ .
- f) Z (the integers) is enumerable. Here is one way to list its members: 0, 1, -1, 2, -2, [What function f enumerates this list? Use a table of values to see the pattern.]
9. The set of positive rational numbers (number expressible as m/n , m, n are positive integers)
Start with the enumeration of ordered pairs of positive integers, and replace the pair (m,n) by the rational number m/n ;
(1,1),(1,2),(2,1),(1,3),...
1/1,1/2,2/1,1/3
10. The set of rational numbers
Take the enumeration of ordered pairs
11. The set of ordered triples of positive integers
12. The set of ordered k -tuples of positive integers, for any fixed k
13. The set of finite sequences of positive integers less than 10
14. The set of finite sequences of positive integers less than b , for any fixed b

15. The set of finite sequences of positive integers
16. The set of finite sets of positive integers
17. Any subset of an enumerable set
18. The union of any two enumerable sets
19. The set of finite strings from a finite or enumerable alphabet of symbols

3 | Theorems and Lemmas

(Thm..3) 1.1 (subsets of an enumerable set are enumerable): If B is enumerable and $A \subset B$, then A is enumerable.

(Thm..4) 1.2 (an enumerable union of enumerable sets yields an enumerable set): If A_1, A_2, A_3, \dots are all enumerable, then so is the set A consisting of the union of all of them. We write this as:

$$A = \bigcup_{i=1}^{\infty} A_i$$

4 | How to show that S is enumerable

1. *Formal Method:* show directly that there is a function $f: P \rightarrow S$ whose range is all of S ;
 - E.g. the even numbers are enumerable: *Proof 1.* $f(n) = 2n$ is a function from P to E whose range is all of E ; *Proof 2.* $g(n) = n$ if n is even and undefined if n is odd
 - Variation of Method: Show that there is an encoding function $G: S \rightarrow P$ that is 1-1. An encoding function $G: S \rightarrow P$ is a 1-1 function that assigns a positive integer to each member of S . A decoding function $g: P \rightarrow S$ is any function that enumerates S . If there is an encoding function G , then there will also be a decoding function, namely the inverse of G . Sometimes the best way to show a set is enumerable is to give an encoding function, rather than a decoding function.
2. *Informal Method:* show that it is possible to place all the members of S in a list in such a way that each member of S will eventually appear, a finite number of positions from the beginning of the list.
 Example: the positive rational numbers. This uses the important technique of placing the entire set in a two-dimensional array, and then putting the members into list form by either moving along successive diagonals or expanding squares.
3. *Appeal to Theorem 1.2:* By appealing to Theorem 1.2: If A_1, A_2, \dots is an enumerable list of enumerable sets, ∞ then the set A consisting of the union of all the A_i (written as $A = \bigcup A_i$) is enumerable.
 Example: the set S of all finite strings of symbols from an enumerable alphabet is enumerable. For if S_i is the set of finite strings of length i , then S_i is enumerable, ∞ and so by the Theorem, since $S = \bigcup S_i$, S is enumerable.