1 | Notions and Theorems Stated (14.1)

Notions and Theorems: Informally Understood

- **Proof Procedure**: series of finite formula with the last designated as the conclusion by rules of inferences or restatements of axioms
- **Why?** To render an implication, if there is one, recognizable by way of obvious steps/rules.
- **Two notions of proof**: i) formal (syntactic) deduction and ii) (semantic) validity;
- a **formal deduction** is a finite array of symbols from Definite, explicit rules. The rules are syntactic because they mention internal structure of formulas and not interpretations.

- *Soundness Thm*: whenever there is a proof of D from Γ, D is a consequence of Γ
- *Completeness Thm*: whenever *D* is a consequence of Γ, there is a proof of D from Γ

Notions and Theorems: As Commonly Understood

Syntactic	deduction (Y)	refutation (?)	demonstration (Y)
Semantic	consequence (?)	unsatisfiability (Y)	validity (?)

- *Soundness Thm*: whenever D is deducible from Γ, D is a consequence of Γ;
- *Godels Completeness Thm*: whenever D is a consequence of Γ, then D is deducible from Γ
- A Note on Decidability:
  - *the consequence relation is not effectively decidable*: there cannot be a procedure, governed by definite and explicit rules, whose application would, in every case, in principle enable one to determine in a finite amount of time whether or not a given finite set Γ of sentences implies a given sentence D.
  - *the consequence relation is effectively semi-decidable* (from the existence of a sound and complete proof procedure): There is a procedure whose application would, in case Γ does imply *D*, in principle enable one to determine in a finite amount of time that it does so.

Notions and Theorems: Boolos, Burgess and Jeffrey's Way
Three Semantic notions in terms of one
(Dfn.1.1) Γ **secures** another set of sentences Δ if every interpretation that makes all sentences in Γ true makes some sentence Δ true. (Note: when the sets are finite, $\Gamma = \{C_1, \ldots, C_m\}$ and = $\{D_1, \ldots, D_n\}$ this amounts to saying that every interpretation that makes $C_1 \wedge \ldots \wedge C_m$ true makes $D_1 \vee \ldots \vee D_n$ true, where the elements of Γ are being taken jointly as premises, but the elements of Δ are being taken alternatively as conclusions, so to speak.)

D is a consequence of Γ	iff	Γ secures $\{D\}$
Γ is unsatisfiable	iff	Γ secures $\emptyset$
D is valid	iff	$\emptyset$ secures $\{D\}$

Three Syntactic notions in terms of one
(Dfn.1.2) the objects of which derivations will be composed are called **sequents**. A sequent $\Gamma \Longrightarrow \Delta$ consists of a finite set of sentences Γ on the left, the symbol $\Longrightarrow$ in the middle, and a finite set of sentences Δ on the right. The sequent is **secure** if its left side Γ secures its right side Δ
(Dfn.1.3) a sequence of sequents (steps) is **a derivation** iff each step is either of the form $\{A\} \Longrightarrow \{A\}$ or follows from earlier steps according to a rules of inference.

A deduction of *D* from Γ is a derivation of $\Gamma \Longrightarrow \{D\}$
A refutation of Γ is a derivation of $\Gamma \Longrightarrow \emptyset$
A demonstration of *D* is a derivation of $\emptyset \Longrightarrow \{D\}$

Central Theorems Restated
(Thm.1.4) 14.1 Theorem (Soundness theorem). Every derivable sequent is secure.
(Thm.1.5) 14.2 Theorem (Godel completeness theorem). Every secure sequent is derivable.

D is deducible from Γ	iff	D is a consequence of Γ
Γ is inconsistent	iff	Γ is unsatisfiable
D is demonstrable	iff	$\{D\}$ is valid

2 Sequent Calculus (Gentzen System)

Table 14-4. *Rules of sequent calculus*

(R0)	$\dfrac{}{\{A\} \Rightarrow \{A\}}$	
(R1)	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}$	Γ subset of Γ', Δ subset of Δ'
(R2*a*)	$\dfrac{\Gamma \cup \{A\} \Rightarrow \Delta}{\Gamma \Rightarrow \{{\sim}A\} \cup \Delta}$	
(R2*b*)	$\dfrac{\Gamma \Rightarrow \{A\} \cup \Delta}{\Gamma \cup \{{\sim}A\} \Rightarrow \Delta}$	
(R3)	$\dfrac{\Gamma \Rightarrow \{A, B\} \cup \Delta}{\Gamma \Rightarrow \{(A \vee B)\} \cup \Delta}$	
(R4)	$\dfrac{\begin{array}{c}\Gamma \cup \{A\} \Rightarrow \Delta \\ \Gamma \cup \{B\} \Rightarrow \Delta\end{array}}{\Gamma \cup \{A \vee B\} \Rightarrow \Delta}$	
(R5)	$\dfrac{\Gamma \Rightarrow \{A(s)\} \cup \Delta}{\Gamma \Rightarrow \{\exists x A(x)\} \cup \Delta}$	
(R6)	$\dfrac{\Gamma \cup \{A(c)\} \Rightarrow \Delta}{\Gamma \cup \{\exists x A(x)\} \Rightarrow \Delta}$	c not in Γ or Δ or $A(x)$
(R7)	$\dfrac{\Gamma \cup \{s = s\} \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$	
(R8*a*)	$\dfrac{\Gamma \Rightarrow \{A(t)\} \cup \Delta}{\Gamma \cup \{s = t\} \Rightarrow \{A(s)\} \cup \Delta}$	
(R8*b*)	$\dfrac{\Gamma \cup \{A(t)\} \Rightarrow \Delta}{\Gamma \cup \{s = t, A(s)\} \Rightarrow \Delta}$	
(R9*a*)	$\dfrac{\Gamma \cup \{{\sim}A\} \Rightarrow \Delta}{\Gamma \Rightarrow \{A\} \cup \Delta}$	
(R9*b*)	$\dfrac{\Gamma \Rightarrow \{{\sim}A\} \cup \Delta}{\Gamma \cup \{A\} \Rightarrow \Delta}$	

14.5 Example. Demonstration of a tautology

(1)	$A \Rightarrow A$	(R0)
(2)	$\Rightarrow A, \sim A$	(R2*b*), (1)
(3)	$\Rightarrow A \vee \sim A$	(R3), (2)

14.6 Example. Refutation of a contradiction

(1)	$\sim A \Rightarrow \sim A$	(R0)
(2)	$\Rightarrow \sim A, \sim\sim A$	(R2*b*), (1)
(3)	$\Rightarrow \sim A \vee \sim\sim A$	(R3), (2)
(4)	$\sim(\sim A \vee \sim\sim A) \Rightarrow$	(R2*a*), (3)
(5)	$A \,\&\, \sim A \Rightarrow$	(4)

3 | Soundness (14.2)

(Thm.3.6) 14.1 Theorem (Soundness theorem). Every derivable sequent is secure.

Proof must i) show that R0 sequent is secure; ii) show that every rule yields a secure sequent i.e. when applied to a secure sequent, the rules yields a secure sequent.

R0 Every R0 sequent $\{A\} \Longrightarrow \{A\}$ is secure ; clearly.

R1 Suppose $\{\Gamma \Longrightarrow \Delta\}$ is secure, where Γ is a subset of Γ' and Δ is a subset of Δ'.
Consider any interpretation that makes all the sentences in Γ' true.
Show: the interpretation makes some sentence in Γ' true.
Since Γ is a subset of Γ', it makes all the sentences in Γ true, and so by the security of $\Gamma \Longrightarrow \Delta$ it makes some sentence in Γ true and, since Γ is a subset of Γ', thereby makes some sentence of Γ true.

R2a Suppose $\Gamma \bigcup \{A\} \Longrightarrow \Delta$ is secure
Consider any interpretation that makes all the sentences in Γ true.
Show: the interpretation makes some sentence in $\{\neg A\} \bigcup \Delta$ true
Either the given interpretation makes A true or not.
If it does, then it makes all the sentences in $\Gamma \bigcup \{A\}$ true, and so by the security of $\Gamma \bigcup \{A\} \Longrightarrow \Gamma$ makes some sentence in Γ true, and so makes some sentence in $\{\neg A\} \bigcup \Gamma$ true.
If it does not, then it makes $\neg A$ true, and so it makes some sentence in $\{\neg A\} \bigcup \Gamma$ true.

R5 Suppose that $\Gamma \Longrightarrow \{A(s)\} \bigcap \Delta$ is secure
Consider any interpretation that makes all sentences in Γ true
Show that: $\Gamma \Longrightarrow \{\exists x A(x)\} \bigcap \Gamma$ is secure
From supposition, the interpretation makes some sentence in $\{A(s)\} \bigcap \Gamma$ true. If the sentence is one in Γ, then clearly the interpretation makes some sentence in $\{\exists x A(x)\} \bigcap \Gamma$ true.
If the sentence is A(s), then the interpretation makes $\exists x A(x)$ true, and so again the interpretation makes some sentence in $\{\exists x A(x)\} \bigcap \Gamma$ true. So, This suffices to show that $\Gamma \Longrightarrow \{\exists x A(x)\} \bigcup \Gamma$ is secure, which is what (R5) requires.

R7 Suppose $\Gamma \bigcap \{s = s\} \Longrightarrow \Gamma$ is secure
Consider any interpretation of a language containing all symbols in Γ and Δ that makes all sentences in Δ true.
Show: $\Gamma \Longrightarrow \Delta$ is secure
If there is some symbol in s not occurring in Γ or Δ to which this interpretation fails to assign a denotation, alter it so that it does. The new interpretation will still make every sentence in Γ true by extensionality, and will make $s = s$ true. By the security of $\Gamma \bigcup \{s = s\} \Longrightarrow \Gamma$, the new interpretation will make some sentence in Γ true, and extensionality implies that the original interpretation already made this same sentence in Γ true. This suffices to show that $\Gamma \Longrightarrow \Delta$ is secure.

- . . . (Similarly for all other rules)

4 | Completeness (14.2)

(Lemma.4.7) $\Gamma \Rightarrow \Delta$ iff $\Gamma \cup \neg\Delta$ is inconsistent

1. If $\{C_1, \ldots, C_m\} \Rightarrow \{D_1, \ldots, D_n\}$ is derivable then
   $\{C_1, \ldots, C_m, \neg D_1\} \Rightarrow \{D_2, \ldots, D_n\}$,
   $\{C_1, \ldots, C_m, \neg D_2\} \Rightarrow \{D_3, \ldots, D_n\}$,
   ...
   $\{C_1, \ldots, C_m, \neg D_n\} \Rightarrow \emptyset$ are derivable by (R2b)
2. Show that the set S of all consistent sets has the satisfiability properties (S0)-(S8).
   By the main lemma of the preceding chapter, in order to show every consistent set is satisfiable, it will suffice to show that the set S of all consistent sets has the satisfiability properties (S0)–(S8). (For any consistent set Γ will by definition belong to S, and what Lemma 13.3 tells us is that if S has the satisfaction properties, then any element of S is satisfiable.)
   - S0 If $\Gamma \implies \emptyset$ is derivable, and Γ_0 is a subset of Γ, then $\Gamma \implies \emptyset$ is derivable.
   - S1 If A and $\neg A$ are both in Γ, then $\Gamma \implies \emptyset$ is derivable.
   - S2 If $\Gamma \implies \emptyset$ is not derivable and $\neg lnegB$ is in Γ, then $\Gamma \bigcup \{B\} \implies \emptyset$ is not derivable.
   - S3 If $\Gamma \bigcup \{B\} \implies \emptyset$ and $\Gamma \bigcup \{C\} \implies \emptyset$ are both derivable, then $\Gamma \bigcup \{B \vee C\} \implies \emptyset$ is derivable.
   - S4 If $\Gamma \bigcup \{B(c)\} \implies \emptyset$ is derivable, where c does not occur in $\Gamma \bigcup \{\exists x B(x)\} \implies \emptyset$, then $\Gamma \bigcup \{\exists x B(x)\} \implies \emptyset$ is derivable
   - S5 If $\Gamma \bigcup \{B(c) \implies \emptyset\}$ is derivable, where c does not occur in
   - S6 If $\Gamma \bigcup \{\neg B(t) \implies \emptyset\}$ is derivable for some closed term t, then $\Gamma \bigcup \{\neg \exists x B(x)\} \implies \emptyset$ is derivable
   - S7 $\neg \bigcup \{t = t\} \implies \emptyset$ is derivable for some closed term t, then $\Gamma \implies \emptyset$ is derivable
   - S8 If $\Gamma \bigcup \{B(t)\} \implies \emptyset$ is derivable, then $\Gamma \bigcup \{B(s), s = t\} \implies \emptyset$ is derivable:

5 | Logic and Mathematics (14.3)

What is the relationship between the formal notion of deduction of a sentence from a set of sentences, and the notion in unformalized mathematics of a proof of a theorem from a set of axioms?

Suppose theorems and axioms of ordinary mathematics are expressed as sentences of formal first-order language (as almost all can; see ch. 10) Then
(Prop.5.8) If there is a deduction in the logician's formal sense of the theorem from the axioms, there will be a proof in the mathematician's ordinary sense (since each formal rule of inference of deduction corresponds to some ordinary mode of argument as used in mathematics)
(Prop.5.9) **Hilbert's Thesis** if there is a proof in the ordinary sense, then there will be a deduction in our very restrictive format.