1 | Size and Number of Models (§12.1)

Dfn **1.1.** *Some Definitions concerning models:*

- *a model of a sentence or set of sentences is an interpretation in which the sentence, or every sentence in the sentence, comes out true*
- Γ *implies D if every model of* Γ *is a model of D; D is valid if every interpretation is a model of D, and* Γ *is unsatisfiable if no interpretation is a model of* Γ.
- *the size of a model is the size of the model domain; it is* finite, denumerable *or* non-denumerable
- *a set of sentence is said to have arbitrarily large finite models if for every positive integer m there is a positive integer* $n \geq m$ *such that the set has a model of size n*

Model Size: Logical Space For a set of sentences Γ, are there Γ with

- no models? **No. Always possible to have a set as domain and interpret constant/predicates. Note: Contradictions have models.**
- only finite models? **No. (LS-Thm)**
- only denumerable models?
- only non-denumerable models? **No. (LS-Thm)**
- finite and denumerable models only? **Yes. (LS-Thm)**
- finite and non-denumerable models only? **No. (LS-Thm)**
- denumerable and non-denumerable models?
- finite, denumerable and non-denumerable models?
- How many *different* models a sentence or set of sentences may have of a given size? There are always a nonenumerable infinity of models if there are any.

(Ex.1.1) *A sentence with models only of a specified finite size.* For each positive integer n there is a sentence In involving identity but no nonlogical symbols such that In will be true in an interpretation if and only if there are at leastndistinct individuals in the domain of the interpretation. Then $J_n = \neg I_{n+1}$ will be true iff there are at most n individuals, and $K_n = I_n \land J_n$ will be true iff there are exactly n individuals.
$\forall x \forall y \exists z (z \neq x \land z \neq y)$
(Ex.1.2) *A sentence with only infinite models.* Let R be a two-place predicate. Then the following sentence A has a denumerable model but no finite models: $\forall x \exists y Rxy \land \forall x \forall y \neg (Rxy \land Ryx) \land \forall x \forall y \forall z ((Rxy \land Ryz) \to Rxz)$
(Ex.1.3) *A sentence with non-denumerable many models.* Consider the empty language, with identity but no non-logical predicates, where an interpretation is just a nonempty set as domain.
$\exists x \forall y (y = x)$ "there is just one thing in the domain"
For any object as you wish, the interpretation whose domain is $\{a\}$, the set whose only element is a, is a model of this sentence. So for each real number we get a model.

2 | Isomorphism (§12.1)

Dfn **2.1.** *Two interpretations P and Q of the same language L are isomorphic iff there is a correspondence (total, one-to-one, onto function) j between individuals p in the domain* $|P|$ *and individuals q in the domain* $|Q|$ *under certain conditions:*

- *for every n-place predicate R and all* $p_1, \ldots, p_n$ *in* $|P|$,
  $R^P(p_1, \ldots, p_n)$ *iff* $R^Q(j(p_1), \ldots, j(p_n))$
- *for every constant c,* $f(c^P) = c^Q$
- *for every n-place function symbol f and all* $p_1, \ldots, p_n$ *in* $|P|$,
  $j(f^P(p_1 \ldots, p_n)) = f^Q(j(p_1), \ldots, j(p_n))$.

(Ex.2.4) **Inverse order and mirror arithmetic**. Consider L with two-place predicate $<$, the interpretation with domain the natural numbers $\{0, 1, 2, 3, \ldots\}$ and with $<$ denoting the usual strict less-than order relation, and by contrast the interpretation with domain the non-positive integers $\{0, 1, 2, 3, \ldots\}$ and with $<$ denoting the usual strict greater-than relation. The correspondence associating n with−nis an isomorphism, since mis less than nif and only if −mis greater than −n, as required by (I1).

Thm 2.2. *Let X and Y be sets, and suppose there is a correspondence j from X to Y. Then if Y is any interpretation with domain Y, there is an interpretation X with domain X such that X is isomorphic to Y. In particular, for any interpretation with a finite domain having n elements, there is an isomorphic interpretation with domain the set* $\{0, 1, 2, ..., n1\}$*, while for any interpretation with a denumerable domain there is an isomorphic interpretation with domain the set* $\{0, 1, 2, ...\}$ *of natural numbers*

Thm 2.3. *Proposition(Isomorphism lemma). If there is an isomorphism between two interpretations P and Q of the same language L, then for every sentence A of L we have* $P \models A$ *iff* $Q \models A$

Cor **2.4.** *Canonical-domains lemma*

(a) Any set of sentences that has a finite model has a model whose domain is the set $\{0, 1, 2, ..., n\}$ *for some natural number n.*
(b) Any set of sentences having a denumerable model has a model whose domain is the set $\{0, 1, 2, \ldots\}$ *of natural numbers.*

3 | The Lowenheim-Skolem and Compactness Theorems (§12.3)

Let Γ be any set of sentences,

Thm 3.1. *Lowenheim-Skolem Thm: If* Γ *has a model, then it has an enumerable model*

Thm 3.2. *Compactness Theorem: If every finite subset of* Γ *has a model, then* Γ *has a model*

Cor **3.3.** *Overspill Principle: If* Γ *has arbitrarily large finite models then it has a denumerable model*

(dfn.3.5) a set Γ of sentences is **implicationally complete** if for every sentence *A* in its language, either *A* or $\neg A$ is a consequence of Γ.
a set Γ of sentences is **denumerably categorical** if two denumerable models of Γ are isomorphic

Cor **3.4.** *Vaught's Test: If* Γ *is a denumerably categorical set of sentences having no finite models, then* Γ *is complete*

Cor **3.5.** *Canonical-domains lemma*

(a) Any Γ *that has a model, has a model whose domain is either the set of natural numbers* $< n$ *for some positive n, or else the set of all natural numbers*
(b) Any Γ *not having function symbols or identity that has a model, has a model whose domain is the set of all natural numbers.*

4 | Significance of L-S Thm and Compactness Thm (§12.3)

Dfn **4.1.** *Let D be a sentence, and* Γ *a set of sentences,*

- *D is defined to be deducible from a finite set* Γ *iff there is a deduction of the sentence from* Γ
- *A deduction from a subset of a set always counts as a deduction from that set itself*
- *D is defined to be deducible from an infinite set* Γ *iff it is deducible from some finite subset.*
- *D is defined to be demonstrable if it is deducible from the empty set of sentences, and a set of sentences* Γ *is defined to be inconsistent if the constant false sentence* $\bot$ *is deducible from it.*

Thm 4.2. *Let D be a sentence, and* Γ *a set of sentences,*

- *Soundness theorem: if D is deducible from* Γ*, then D is a consequence of* Γ
- *Godel Completeness theorem: if D is a consequence of* Γ*, then D is deducible from* Γ
- *From Completeness it follows that if D is valid, then D is demonstrable, and that if* Γ *is unsatisfiable, then* Γ *is inconsistent).*
- *Soundness and Completeness => Compactness*
- *Compactness => LS Theorem*
- *Soundness and Completeness can be proven independently of compactness*