1 | Interpretation

Dfn **1.1.** *An interpretation M for a language L is:*

1. *A non-empty domain/universe of discourse | M | of objects.*
2. *A denotation function for each non-logical symbol of L.*
   (a) *Constants. For each constant c, the denotation c^M is to be some individual in domain $|M|$.*
   (b) *Predicates. For each n-place predicate R, an n-place relation R^M on $|M|$.*
   (c) *Function symbols. For each n-place function symbol f, an n-place function fM from $|M|$ to $|M|$.*

2 | Truth under an Interpretation

Dfn **2.1.** *True under an Intrepretation: $M \models F$*
read: "M satisfies F"; "M makes F true"; "F is true on the interpretation M."
Rem **2.2.** *This symbol, $\models$, is NOT part of the language L. It's really just an abbreviation for 'makes true'. We define this inductively on the sentences (closed formulas) of L.*

Dfn 2.3. *Atomic SentencesM* $\models R(t_1, \ldots, t_n)$ iff $R^M(t_1^M, \ldots, t_n^M)$

Rem 2.4. The atomic sentence is true in the interpretation just in case the relation that the predicate is interpreted as denoting hold of the individuals that the constants are interpreted as denoting.

Rem 2.5. If function symbols are absent, the closed terms ti are just constants and the denotations are well-defined as part of the interpretation. If function symbols are present, both (1a) and (1b) are still fine, but we need to define the denotation tM of a closed term t as follows (the denotation will always be an object in |M|)

Dfn 2.6. *Terms*
If t is a constant c, then $t^M = c^M$ is already defined.
Suppose t_1^M, t_2^M, …, t_n^M are defined and f is an n-place function symbol. Then $(f(t_1,\ldots t_n))^M = f^M(t_1^M, \ldots, t_n^M)$

Dfn 2.7. Complex Sentences (Connectives Only)
$M \models \neg F$ iff not $M \models F$
$M \models F \wedge G$ iff $M \models F$ and $M \models G$

Rem 2.8. The truth conditions for $\vee$, $\rightarrow$, $\leftrightarrow$are abbreviations.

Abv 2.9. $M \models F \wedge G$ iff not $M \models F$ or not $M \models$G

Abv 2.10. $M \models F \wedge G$ iff not $M \models F$ or not $M \models G$

3 | Quantified Sentences

For quantified sentences, it is more complicated. There are roughly three different approaches to quantification: substitutional, objectual and plural. In the first two cases, quantifiers range over objects, the third, which we omit here ranges over.

Dfn **3.1.** *Substitutional Approach*
$M \models \forall x F(x)$ iff for every closed term (constant) t in L(FOL), $M \models F(t)$ $M \models \exists x F(x)$ iff for some closed term t, $M \models F(t)$
A universal quantification is true iff every substitutional instance is true, and existential quantification just in case some substitutional instance is true.

Rem 3.2. The idea is that a quantified sentence is true just in case all of the formulas that result from substituting a constant.

Rem 3.3. (BBJ, pg. 116)Simple, tempting and wrong: it produces results not in agreement with intuition, unless it happens that every individual in the domain of the interpretation is denoted by some term of the language. If the domain of the interpretation is enumerable, we could always expand the language to add more constants and extend the interpretation so that each indivdiaul in the domain is the denotation of one of them. But, we cannot do this when the is non-enumerable. (At least we cannot do so while continuing to insist that a language is supposed to involve only a finite or enumerable set of symbols. Of course, to allow a 'language' with a nonenumerable set of symbols would involve a considerable streching of the concept.

Dfn **3.4.** *Objectual Approach 1*
$M \models \forall x F[m]$ *iff* $M^c_m \models F(c)$

Dfn 3.5. $M \models F[m]$ means 'if we considered the

Dfn 3.6. $M \models \forall x F(x)$ iff for every m in the domain, $M \models F[m]$

Dfn 3.7. $M \models \forall x F(x)$ iff for some m in the domain, $M \models F[m]$

Rem 3.8. (BBJ, pg. 117): We should not attempt to extend the given language L so as to provide constants for every individual in the domain of the interpretation at once. In general, that cannot be done without making the language nonenumerable. However, if we consider any particular indvidual int the domain, we could extend the language and interpreation to give just it a name, and what we do in defining when $M \models F[m]$, to mean 'if we consdered the extended language $L \cup c$ obtained by adding a new constant c in to our given language L, and if among allt the extensions of our given interpretation M to an interpretation of this extended language we considered the one M^c_m that assigns c to the denotation m then $F(c)$ would be true: $M^c_m \models F(c)$. (For definiteness, let us say the constant to be added should be the first constant not in L in our fixed enumeration of the stock of constants.)

Rem 3.9. Still, doesn't this observation illustrate the expressive limit of first order logic, namely that either we need an infinite vocabulary — which no language has and no human could understand since there they are finite — or we need an infinite number of domain that again have to be understood individually for all objects in the domain to be represented in the language.

Rem 3.10. Problem: We want to guarantee that F holds of every object in the domain; and there may be objects in the domain that are not named by any closed term.

Dfn 3.11. Shift Interpretation: If c is a constant and m is in M, then M* = Mc m is the interpretation which is just like M except that cM* = m. [The constant c could be new or already part of L.] Def. $M \models F[m]$ (m satisfies F(x)) iff $Mc, m \models F(c)$. This can be extended to formulas with more than one free variable. With this idea, we can define the clauses for $\forall x$ and $\exists x$

Rem 3.12. For every object m in the domain, if we extend the language by adding new constant c and extend the interpretation to make c denote m, the extended interpretation makes F(c) true.

Dfn **3.13.** *Objectual Approach 2:* $M \models \forall x Fx$ *if and only if for every* m *in the domain* $|M|$, $M \models F[m]$
$M \models \exists x Fx$ *if and only if for some* m *in the domain* $|M|$, $M \models F[m]$

4 Metalogical Notions

- A set of sentences Γ **implies** or has a **consequence** the sentence D if there is no interpretation that makes every sentence in Γ true, but makes D false
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