Proofs in Propositional Logic

PHIL2100
George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

1 Rules of Implication - 7.1, 7.2

- In chapter 6, we used truth tables to determine validity or invalidity of arguments
- In chapter 7, we study proofs in propositional logic, which are also used to determine **logical validity**, **logical inconsistency**, **logical truth**, **logical falsehood**

Dfn 1.1. a **rule of inference** consists of an argument form whose premises imply the conclusion

Some Rules of Inference

- **Modus Ponens (MP)**

1	$p \supset q$
2	p
3	q

- **Modus Tollens (MT)**

1	$p \supset q$
2	$\sim q$
3	$\sim p$

- **Pure Hypothetical Syllogism (HS)**

1	$p \supset q$
2	$q \supset r$
3	$p \supset r$

- **Disjunctive Syllogism (DS)**

1	$p \vee q$
2	$\sim p$
3	q

Proofs in Propositional Logic

PHIL2100
George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

- **E.g. 1**:

1	N	
2	$N \supset K$	
3	$N \vee F$	
4	K	1, 2 MP

**Note that you do not need to use all the given premises in a proof.

- **E.g. 2**:

1	$\sim (E \vee F)$	
2	$(E \vee F) \vee (N \supset K)$	
3	$(N \supset K)$	

- **E.g. 3**:

1	$\sim\sim (E \vee F)$	
2	$\sim (E \vee F) \vee (N \supset K)$	
3	$(N \supset K)$	1, 2 DS

- **E.g. 4**:

1	$(E \vee F)$	
2	$\sim (E \vee F) \vee (N \supset K)$	
3	$(N \supset K)$	1, 2 DS *****Incorrect *****

Note: DS rule works only on a compound statement that is a disjunction and a negation; in this case there is no negation.

George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

- **E.g.5**:

1	$J \supset (K \supset L)$
2	$L \vee J$
3	$\sim L$
4	...
5	$\sim K$

- **E.g. 6**:

1	$\sim G \supset (G \vee \sim A)$
2	$\sim A \supset (C \supset A)$
3	$\sim G$
4	...
5	$\sim C$

- **E.g. 7**:

1	$H \supset [\sim E \supset (C \supset \sim D)]$
2	$\sim D \supset E$
3	$E \vee H$
4	$\sim E$
5	...
6	$\sim C$

- **E.g. 8**:

1	$(R \supset F) \supset [(R \supset \sim G) \supset (S \supset Q)]$
2	$(Q \supset F) \supset (R \supset Q)$
3	$\sim G \supset F$
4	$Q \supset \sim G$
5	...
6	$S \supset F$

Proofs in Propositional Logic

PHIL2100
George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

- **Modus Ponens (MP)**: $p \supset q, p // q$
- **Modus Tolens (MT)**: $p \supset q, \sim q // \sim p$
- **Hypothetical Syllogism (HS)**: $p \supset q, q \supset r // p \supset r$
- **Disjunctive Syllogism (DS)**: $p \vee q, \sim p // q$
- **Constructive Dilemma (CD)**

1	$(p \supset q) \bullet (r \supset s)$
2	$p \vee r$
3	$q \vee s$

- **Simplification (Simp)**

1	$p \bullet q$
2	p

- **Conjunction (Conj)**

1	p
2	q
3	$p \bullet q$

- **Addition (Add)**

1	p
2	$p \vee q$

- **E.g. 1**: $(P \supset R) \supset (M \supset P)$, $(P \vee M) \supset (P \supset R)$, $P \vee M$, so $R \vee P$
- **E.g. 2**: $(\sim H \supset (\sim T \supset R)$, $H \vee (E \supset F)$,$\sim T \vee E$, $\sim H \bullet D$, so $R \vee F)$
- **E.g. 3**: $(R \supset H) \bullet (S \supset I)$, $(\sim H \bullet \sim L) \supset (R \vee S)$, $\sim H \bullet (K \supset T)$, $H \vee \sim L$, so, $I \vee M$
- **E.g. 4**: $(S \supset Q) \bullet (Q \supset \sim S), S \vee Q, \sim Q$, so, $P \bullet R$

George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

2 Rules of Replacement - 7.3, 7.4

- **De Morgan's Rule (DM)**
  $\sim (p \bullet q) :: (\sim p \vee \sim q)$
  $\sim (p \vee q) :: (\sim p \bullet \sim q)$
- **Commutativity (Com)**
  $(p \vee q) :: (q \vee p)$
  $(p \bullet q) :: (q \bullet p)$
- **Associativity (Assoc)**
  $[p \vee (q \vee r)] :: [(p \vee q) \vee r]$
  $[p \bullet (q \bullet r)] :: [(p \bullet q) \bullet r]$
- **Distribution (Dist)**
  $[p \bullet (q \vee r)] :: [(p \bullet q) \vee (p \bullet r)]$
  $[p \vee (q \bullet r)] :: [(p \vee q) \bullet (p \vee r)]$
- **Double Negation (DN)**
  $p :: \sim\sim p$

- **Ex. 7.3.III.14**: $\sim (J \vee K)$, $B \supset K$, $S \supset B$ // $\sim S \bullet \sim J$
- **Ex. 7.3.III.15**: $(G \bullet H) \vee (M \bullet G)$, $G \supset (T \bullet A)$, // A
- **Ex. 7.3.III.26**: $A \bullet (F \bullet L)$, $A \supset (U \vee W)$, $F \supset (U \vee X$ // $U \vee (W \bullet X)$

Proofs in Propositional Logic

PHIL2100
George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

3 More Rules of Replacement (Ch. 7.4)

- **Transportation (Trans)**:
  $(p \supset q) :: (\sim q \supset \sim p)$
- **Material Implication (Impl)**:
  $(p \supset q) :: (\sim p \vee q)$
- **Material Equivalence (Equiv)**:
  $(p \equiv q) :: (p \supset q) \bullet (q \supset p)$
  $(p \equiv q) :: (p \bullet q) \vee (\sim p \bullet \sim q)$
- **Exportation (Exp)**
  $(p \bullet q) \supset r :: (p \supset (q \supset r)$
- **Tautology (Taut)**
  $p :: (p \vee p)$
  $p :: (p \bullet p)$

4 Conditional Proof (Ch. 7.5)

The schema for the conditional proof (ACP) is as follows.

1	...	
2	...	
3	...	
4	| P	ACP
5	| ...	
6	| Q	
7	$P \supset Q$	4-7, CP
8	...	

George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

5 All Rules of Propositional Logic

$p \supset q, p // q$	**Modus Ponens (MP)**
$p \supset q, \sim q // \sim p$	**Modus Tolens (MT)**
$p \supset q, q \supset r // p \supset r$	**Hypothetical Syllogism (HS)**
$p \vee q, \sim p // q$	**Disjunctive Syllogism (DS)**
$(p \supset q) \bullet (r \supset s), p \vee r, // q \vee s$	**Constructive Dilemma (CD)**
$p \bullet q // p$	**Simplification (Simp)**
$p, q // p \bullet q$	**Conjunction (Conj)**
$p // p \vee q$	**Addition**
$\sim (p \bullet q) :: (\sim p \vee \sim q)$	**De Morgan's Rule (DM)**
$\sim (p \vee q) :: (\sim p \bullet \sim q)$	
$(p \vee q) :: (q \vee p)$	**Commutativity (Com)**
$(p \bullet q) :: (q \bullet p)$	
$[p \vee (q \vee r)] :: [(p \vee q) \vee r]$	**Associativity (Assoc)**
$[p \bullet (q \bullet r)] :: [(p \bullet q) \bullet r]$	
$[p \bullet (q \vee r)] :: [(p \bullet q) \vee (p \bullet r)]$	**Distribution (Dist)**
$[p \vee (q \bullet r)] :: [(p \vee q) \bullet (p \vee r)]$	
$p :: \sim\sim p$	**Double Negation (DN)**
$(p \supset q) :: (\sim q \supset \sim p)$	**Transportation (Trans)**
$(p \supset q) :: (\sim p \vee q)$	**Material Implication (Impl)**
$(p \equiv q) :: (p \supset q) \bullet (q \supset p)$	**Material Equivalence (Equiv)**
$(p \equiv q) :: (p \bullet q) \vee (\sim p \bullet \sim q)$	
$((p \bullet q) \supset r) :: (p \supset (q \supset r))$	**Exportation (Exp)**
$p :: (p \vee p)$	**Tautology (Taut)**
$p :: (p \bullet p)$	
. . . (see above)	**Conditional Proof (ACP)**

Proofs in Propositional Logic

PHIL2100
George Belic (belic@ufl.edu)
http://www.clas.ufl.edu/users/belic/

6 Indirect Proof - 7.6

A Famous Proof by Contradiction Euclid's proof (4th century B.C.) showed that there is no greatest prime number. A prime is defined as a number with exactly two divisors. (e.g. 3 is a prime, because it can be evenly divided by 3 and 1 only)

1	Suppose there were only finitely many primes.
2	Let p denote the largest prime.
3	Let q be the product of the first p numbers.
4	Then q + 1 is not divisible by any of the first p numbers.
5	Then q + 1 is also prime and is greater than p.
6	Contradiction.
7	So, there are infinitely many primes.

7 Proving Logical Properties of Statements

We can show that

- A statement p is **Logically true** just in case p can be proven from no premises.
- A statement p is **Logically false** just in case $\sim p$ can be proven from no premises.
- A set of statements is **Logically Inconsistent** just in case if each statement is a premise in an argument then a contradiction can be proven ($p \bullet \sim p$)
- A number of logical properties **cannot** be shown using proofs: i) that an argument is invalid; ii) that a set of statements are consistent; iii) that a statement is logically contingent

1	$L \supset [\sim M \supset (N \bullet O)]$
2	$\sim N \bullet P$
3	...
4	$(L \supset (M \bullet P)$